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The topology of interfaces

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Abstract. A discrete model of an interface separating two phases is considered. The interface is defined as the boundary of a self-avoiding n-omino with volume n embedded in the hypercubic lattice. We prove that if there is no surface tension in the interface, then it has 'non-trivial' topology with probability one in the scaling limit. Here non-trivial topology means that the interface will consists of several disjoint components and will have non-zero genus. Moreover, the total area of the n-omino will be proportional to its volume.

1. Introduction

Interfaces separating two phases in statistical systems such as the Ising model have received considerable attention in recent years. Traditionally, self-avoiding surfaces (Binder 1979, Eguchi and Kawai 1982, Fröhlich 1985, Nelson 1988, Privman and Švrakić 1988) were used to model interfaces, and some, like the solid-on-solid model, proved very useful in simulating important properties (like the roughening transition) of an interface (Weeks 1980). In this paper we shall develop an alternative way of looking at this problem. We begin by defining our model.

Let $\mathcal{Z}^d$ be the d-dimensional hypercubic lattice. We can represent the vertices of $\mathcal{Z}^d$ by d-tuples $v = (v_1, v_2, \ldots, v_d)$. Let $\{e_i\}_{i=1}^d$ be the set of orthogonal unit vectors with endpoints in $\mathcal{Z}^d$. Then $e_i \cdot e_j = \delta_{ij}$. An edge with vertices in $\mathcal{Z}^d$ can be represented by a double (v, e_i), and has endpoints v and $v + e_i$. We can use a triple (v, e_i, e_j) to represent a unit square (plaquette) with vertices v, $v + e_i$, $v + e_j$ and $v + e_i + e_j$ in $\mathcal{Z}^d$. In general, a $(q+1)$-tuple $(v, e_{i_1}, e_{i_2}, \ldots, e_{i_q})$, with all the i_j distinct, represents a q-dimensional hypercube, or q-cell, in $\mathcal{Z}^d$. A q-cell is therefore the interior and boundary of $I^q = I \times I \times \ldots \times I$, where $I = [0,1] \subset \mathcal{R}$, with vertices in $\mathcal{Z}^d$.

In this paper we are interested in a very specific case, the embeddings of d-cells in $\mathcal{Z}^d$. The boundary of a d-cell consists of $2d$ faces, which are $(d-1)$-cells and which we call the *faces* of the d-cell. One way to create a 2-omino from two d-cells is by identifying two faces, one on each d-cell. A third d-cell can be added to this 2-omino, by again identifying two faces, one on the cell to be added, and one on the 2-omino. Let the set of all n-ominoes, $\mathcal{Q}_n^d$, be the set of all subgraphs of $\mathcal{Z}^d$ made from n d-cells, where at most two d-cells meet at every face. We say that two d-cells are *joined* if they

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share the same face, and an n-omino is *connected* if every two d-cells in the n-omino are elements in a sequence of d-cells such that neighbouring d-cells are joined. The n-omino in figure 1(a) is not connected, but in figure 1(b) we illustrate a connected n-omino. We call a q-cell in the n-omino *common* if all the d-cells on which it is incident form a connected sub-n-omino. Since every face in the n-omino is incident on either one or two d-cells, all faces are common. However, not all q-cells, with $q < d-1$ need be common: in figure 1(a) we chose $d = 3$; the edge (1-cell) marked e and the vertex (0-cell) marked v are not common, but the edge e' and the vertex v' are common. (Usually, the term n-omino is used to refer only to the two-dimensional case. In this paper we extend the definition to d dimensions (see Klarner 1965, 1967, Klarner and Rivest 1973, 1974).)

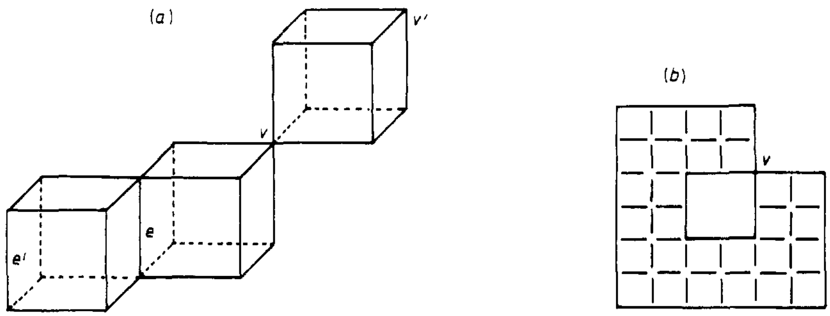

Figure 1. The n-omino in (a) is not connected and contains q-cells which are not common. e and v are an edge and a vertex which are not common. On the other hand, the edge and the vertex, e' and v' respectively, are common. The n-omino in (b) is connected, but not self-avoiding. The vertex v is not common.

A connected n-omino is *self-avoiding* if every face on the n-omino is incident on at most two d-cells, and if every q-cell, $0 \leq q \leq d-1$, in the n-omino is common. The n-omino in figure 1(b) is connected, but not self-avoiding (the vertex marked v is not common). In the rest of the paper we shall mean *self-avoiding* n-omino whenever we say n-omino. Let $\mathcal{S}_n$ be the set of all self-avoiding n-ominoes made from n d-cells. Let $\sigma \in \mathcal{S}_n$. The boundary of σ consists of all the faces (($d-1$)-cells) in σ which are incident on only one d-cell in the n-omino. Let $\partial\sigma$ be the boundary of σ, where we choose ∂ to be the homology boundary operator. Since $\partial\sigma$ separates the 'inside' of σ from the 'outside', we call it an *interface*. Clearly, $\partial\partial\sigma = 0$, that is the boundary of σ has no boundary. An example of an interface is the boundary components of a punctured disc in two dimensions (figure 2).

A self-avoiding n-omino in d dimensions is one possible generalization of polyominoes to higher dimensions, which have received much attention in the literature (Eden 1961, Read 1962, Klarner 1965, 1967). In higher dimensions there are interesting topological considerations to be taken into account, and we shall consider some of those in this paper. In this paper we also aim to generalize the rigorous results on surfaces in our previous works (Janse van Rensburg and Whittington 1989, 1990, hereafter referred to as I and II respectively). n-ominoes are a generalization of self-avoiding surfaces, and since the interfaces of n-ominoes constructed from three-cells are self-avoiding surfaces, they provide a fresh look at an old problem. The methods we use were inspired by those in I and II, and by the remarkable work of Madras (1989) on a pattern theorem for lattice animals and trees.

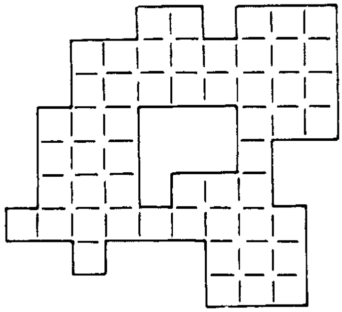

Figure 2. A punctured disc in two dimensions.

n-ominoes also model physical systems studied in the laboratory. Consider for example a microemulsion (Langevin *et al* 1982), which forms in a mixture of oil, water and surfactants. The surfactant molecules have hydrophylic and hydrophobic ends opposite each other. The surfactant molecules therefore separate the oil from the water. Langevin *et al* (1982) studied such a system and identified several phases. The n-ominoes we defined model such a system in the high-temperature, and low surface tension and curvature energy, regime (see Huse and Leibler 1988). Under these conditions, Langevin *et al* (1982) identified a 'random isotropic phase', where the microemulsion has non-trivial topology. In this paper we investigate the possibility that this phase exists in our model (see also Leibler 1989).

Frölich (1985) posed an interesting question: is there a regime in the three-dimensional Ising model where the interface has non-trivial genus with probability 1, in the scaling limit? If we identify the +1 spins with the cells in the n-omino, and the -1 spins with the rest of $\mathcal{Z}^3$, then we have a correspondence between the Ising model and n-ominoes. The interface between +1 spins and -1 spins in the Ising model corresponds to interfaces in our model. In the high-temperature regime we believe that a random isotropic phase similar to that in microemulsions exists in this model.

In section 2.1 we prove that the number of n-ominoes consisting of n cells is bounded exponentially in n: there exists a constant K such that $s_n < K^{n-1}$, where s_n is the cardinality of $\mathcal{S}_n$. We define patterns and consider their properties in section 2.2. This leads to the definitions of *proper* and *extraordinary* patterns. Let $\mathcal{S}_n(P^g)$ be the set of all n-ominoes containing the pattern P exactly g times. In section 2.3 we prove that there exists a β such that $\lim_{n\to\infty} s_n^{1/n} = \beta$, and, if P is an extraordinary pattern, then there exists a $\beta_0(P)$ such that $\lim_{n\to\infty} s_n(P^0)^{1/n} = \beta_0(P)$. In section 2.4 we consider the incidence of proper patterns in an n-omino. In particular, if P is a proper pattern, then we generalize a lemma in I to prove that

$$\binom{\lfloor n/C \rfloor}{g} s_n(P^0) \leq K^g s_{n+kg}(P^g) \tag{1.1}$$

for C, K and k constants only dependent on d and P.

In section 3 we consider the growth constants of n-ominoes, and the effects that the absence of a chosen pattern has on them. In particular, if we define

$$\psi^P(\epsilon) = \limsup_{n\to\infty} s_n(\epsilon P)^{1/n} \tag{1.2}$$

where $s_n(\epsilon P)$ is the number of n-ominoes which express the proper pattern P precisely $\lfloor \epsilon n \rfloor$ times on their boundaries, then we prove that there exists an $\epsilon > 0$ such that $\psi^P(0) < \psi^P(\epsilon)$. If $\langle N_n(P) \rangle$ is the mean number of times that the proper pattern P is expressed on an n-omino, then this result implies that

$$\liminf_{n\to\infty} \frac{\langle N_n(P)\rangle}{n} = \kappa > 0. \tag{1.3}$$

Therefore, any pattern P will appear at least κn times on average in an n-omino. This result has some immediate consequences. It implies that an interface will have an area proportional to its volume, that any n-omino will contain at least $O(n)$ cavities (thus, the interface is disconnected and consists of $O(n)$ parts). More interesting, the total genus (number of handles in three and more dimensions) of the interface is $O(n)$. We conclude the paper with a few remarks and comments in section 4.

2. Preliminaries and constructions

We start with a few preliminary definitions. Let $\sigma \in \mathcal{S}_n$ be an n-omino. Let $\mathcal{V}$ be the set of all vertices in σ. We define the *top vertex* and the *bottom vertex* of σ by a lexicographic ordering of the vertices in $\mathcal{V}$. To find the *top face* and *bottom face* of σ we need the following definition.

Definition 2.1. A q-cell $(v, a_1, a_2, \ldots, a_q)$, where $a_i = \pm e_j$ for each i and some j, is *perpendicular* to the unit vector e_k if and only if $a_i \cdot e_k = 0$ for all $1 \le i \le q$.

The top vertex t (or bottom vertex b) of σ are incident on d faces ($(d-1)$-cells) $(t, -e_1, -e_2, \ldots, -e_{j-1}, -e_{j+1}, \ldots, -e_d)$ for $1 \le j \le d$ (or $(b, e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_d)$). Only one of these faces is perpendicular to e_1, namely when $j = 1$. We call this the *top face* of the n-omino σ (or the *bottom face*). By definition of t and b, the top and bottom faces are each incident on only one d-cell in the n-omino.

2.1. An exponential bound on s_n

In this section we prove that the cardinality s_n of the set $\mathcal{S}_n$ of all n-ominoes consisting of n d-cells is bounded exponentially in n. We generalize the approach for self-avoiding surfaces in I, which is based on a method of Eden (1961) and Klarner (1967) (applied to n-ominoes in two dimensions by Read (1962)). This procedure has the advantage that it leads to a sequence of improved bounds on s_n, if the n- ominoes are viewed as a sequence of *twigs* (Klarner and Rivest 1973).

Lemma 2.2. Let $d \ge 2$. Then there exists a finite positive constant K such that

$$s_n \le K^{n-1}$$

where $\log K = (2d+1)\log(2d+1) - (2d-1)\log(2d-1)$.

Proof. Suppose that $\sigma \in \mathcal{S}_n$. Let P be the set of all d-cells in σ. Each cell $(v, e_1, e_2, \ldots, e_d)$ in P has centre coordinates $c = v + \frac{1}{2}\sum_j e_j$. The top and bottom cells are found by a lexicographic ordering of the cells in P according to their centre coordinates. Label the bottom cell with the integer 1. We can now label all the cells in P with 2, 3, …, in the following manner. Consider the cell labelled 1. It has $2d$ faces which may be incident on unlabelled cells. Label these cells with 2, 3, …, by ordering them lexicographically (by considering their centre coordinates). Once this has been done, consider the unlabelled cells incident on the cell labelled 2, and label them in the same manner, until all the cells have been labelled to n.

Consider now the cell labelled by p. To each of its $2d$ faces we assign a binary digit. If there is a cell labelled r incident on a face of p, and $r > p$, then the binary digit has value 1. Otherwise it is 0. Hence, each of the cells in the n-omino is now characterized by $2d$ binary digits, each digit corresponding to one face of the cell. Since a digit is 1 only if the face to which it corresponds is incident on a cell with larger label, there will be precisely $(n-1)$ instances where the digit will be 1, if we consider all the cells in the n- omino. The n-omino can now be represented by a string of $2d(n-1)$ binary digits, $(n-1)$ of which are 1s (note that the range of p is from 1 to $(n-1)$; if $p = n$, then there are no d-cells with labels larger than p incident on the d-cell labelled n). The number of ways we can choose $(n-1)$ from $2d(n-1)$ is an upper bound on the number of n-ominoes with n cells. Thus

$$s_n \leq \binom{2d(n-1)}{(n-1)} \leq K^{n-1}$$

where $K = (2d+1)^{(2d+1)}/(2d-1)^{(2d-1)}$, as can be shown (Feller 1950). □

2.2. Patterns

In this section we consider the properties of patterns on n- ominoes. The boundary of any n-omino in d dimensions is a set of $(d-1)$- cells. The *orientation* (in the homological sense) of any of these $(d-1)$-cells are induced by the d-cells on which they are incident and by the homology boundary operator ∂.

Definition 2.3. Let $\sigma \in \mathcal{S}_n$. Then the boundary of σ is an interface consisting of $(d-1)$-cells embedded in $\mathcal{Z}^d$. Let P be a set of $(d-1)$-cells with their orientations in $\partial\sigma$ for some $\sigma \in \mathcal{S}_n$. Then P is a pattern if P is connected and if P consists of at least one oriented $(d-1)$-cell.

Two patterns are identical if they can be superimposed by a translation or a rotation (but not a reflection) of $\mathcal{Z}^d$, and if every two of the superimposed $(d-1)$-cells have the same orientations. The patterns ρ in figure 3(a) and ρ' in 3(b) can be superimposed, but they are not identical, since they do not have the same orientations. Suppose that $\sigma \in \mathcal{S}_n$. Then we say that $\partial\sigma$ contains the pattern P g times if there exists g *disjoint* copies of P in $\partial\sigma$. Let $\mathcal{S}_n(P^g) \subset \mathcal{S}_n$ be the set of n-ominoes such that each $\sigma \in \mathcal{S}_n(P^g)$ contains P exactly g times in its boundary. Let the cardinality of $\mathcal{S}_n(P^g)$ be $s_n(P^g)$. Let

$$\mathcal{M}_i(P) = \bigcup_{n>0} \mathcal{S}_n(P^i).$$

Then $\mathcal{M}_i(P)$ is the set of all n-ominoes which contain P exactly i times in their boundaries. We call the set $\mathcal{M}_1(P)$ the *minimal set* of P.

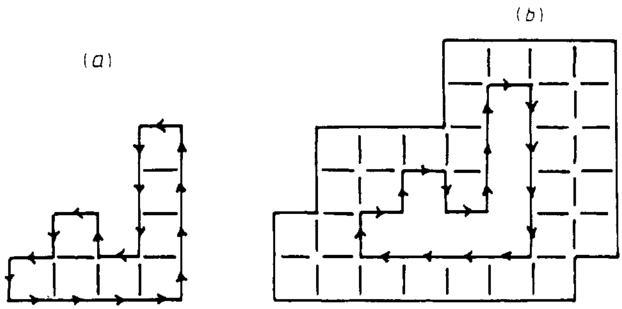

Figure 3. The pattern ρ in (a) and ρ' in (b) can be superimposed, but they are not identical, since they do not have the same orientation.

Definition 2.4. If $\mathcal{M}_1(P) = \emptyset$ and if there exists a finite integer i for which $\mathcal{M}_i(P)$ is not empty, then we call P an *incomplete* pattern. If $\mathcal{M}_1(P) \neq \emptyset$, then P is a *complete* pattern.

For example, if P is a single face on a d-cell σ, then P is an incomplete pattern, since $\partial\sigma$ contains $2d$ disjoint copies of P. Suppose that P is an incomplete pattern and that $i > 1$ is the smallest integer for which $\mathcal{M}_i(P)$ is not empty. Let $\sigma \in \mathcal{M}_i(P)$. Then $\partial\sigma$ contains i (disjoint) copies of the pattern P. Label these patterns P_j, for $1 \leq j \leq i$. All these patterns must be on the same boundary component of σ. To see this, suppose that this is not the case, and that there exists a boundary component of σ which contains P k times, where, $1 < k < i$. There are two possibilities. The boundary component can be a cavity inside σ, or it may be the component separating the inside of σ from a point at infinity. If it is the former, then we fill all the other cavities with d- cells. Let ρ be an n-omino with one boundary component such that $\partial\rho$ does not contain P, and such that we can cover each d- cell of σ with a d-cell of ρ if we superimpose them. Occupy every location outside σ, but inside ρ with a d-cell. Then the new n-omino has two boundary components, and it contains P precisely k times. This is a contradiction, since i is the minimum integer for which this is possible. On the other hand, if P is expressed k times on the 'outside' boundary component of σ, then we simply fill all the cavities inside σ with d-cells. We are then left with an n-omino which expresses P k times on its single boundary component, also a contradiction. Therefore, the i disjoint patterns P must be on the same boundary component.

The *completion* of P with respect to σ, $\bar{P}$, is defined as that pattern which contains the minimum number of faces on a component of $\partial\sigma$ such that all the P_i on σ are contained in $\bar{P}$. $\bar{P}$ is a complete pattern, since $\mathcal{M}_1(\bar{P}) \neq \emptyset$. Every incomplete pattern is therefore a subset of a complete pattern. For example, if P is a single face on a d-cell σ, then $\bar{P} = \partial\sigma$.

Definition 2.5. Let P be a complete pattern and let the minimal set of P be $\mathcal{M}_1(P)$. If there exists a $\sigma \in \mathcal{M}_1(P)$ such that the bottom face of σ is not on P, then P is a *proper pattern*. Otherwise we call P *improper*.

Examples of patterns are given in figure 4. Figures 4(a) and 4(b) are examples of proper patterns in three dimensions, but the pattern in figure 4(c) is improper. Let P be a proper pattern and let

$$n_0 = \min_{\sigma}\{n \mid \sigma \in \mathcal{S}_n(P^1) \text{ and the bottom face of } \sigma \text{ is not on } P\}. \quad (2.1)$$

(That is, we need at least n_0 d-cells to construct an n-omino σ in the minimal set of P such that the bottom cell of σ is not on P.)

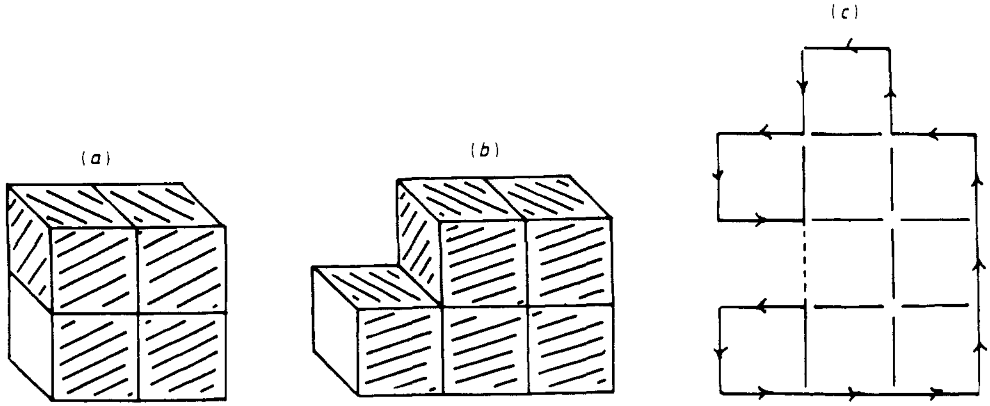

Figure 4. The shaded two-cells on the n-omino in (a) is an extraordinary pattern, and in (b) we have a proper pattern which is not extraordinary. The oriented edges on the boundary of the disc in (c) is an improper pattern.

Definition 2.6. Let $\mathcal{A}$ be the subset of $\mathcal{S}_{n_0}(P^1)$ which has a bottom cell not on P, and where n_0 is defined in equation (2.1). Let the vertex set of an n_0-omino σ in $\mathcal{A}$ be $\mathcal{V}(\sigma)$. Then the *size* of the pattern P is defined by

$$\lambda(P) = \max_{1\leq i\leq d}\{|v_i - w_i| \big| \forall v, w \in \mathcal{V}(\sigma) \text{ and } \forall\sigma \in \mathcal{A}\}.$$

For the rest of this paper we limit our discussion to proper patterns. Note that if a pattern P is on an n-omino with a bottom face not on P, then it is easy to construct a top face not on P, since the n-omino has a finite size. Therefore, if P is a proper pattern, then there exists a $\sigma \in \mathcal{S}_n(P^1)$, for some n only dependent on P, such that the top cell and the bottom cell of σ is not on P.

There are some patterns which we cannot construct by concatenating a second n-omino onto the top face (or bottom face) of any given n-omino, such that the top face is the only $(d-1)$-cell common to the two n- ominoes. We are therefore led to the following definition.

Definition 2.7. Suppose that P is a proper pattern. We call P an *extraordinary pattern* if and only if for any $\sigma \in \mathcal{S}_n(P^1)$ and any $n > 0$ (such that $\mathcal{S}_n(P^1) \neq \emptyset$), there does not exist a $(d-1)$-cell in σ which is incident on two d-cells and which separates σ and P into two disjoint n-ominoes.

The pattern in figure 4(a) is extraordinary, but in figure 4(b) we illustrate a pattern which is not extraordinary.

2.3. Concatenation

Suppose that $\sigma \in \mathcal{S}_n$ and $\rho \in \mathcal{S}_m$. Then we can concatenate σ and ρ by identifying the top face of σ with the bottom face of ρ to form a new n-omino $\sigma \oplus \rho$ consisting of $(n+m)$ d-cells (where the operation $\alpha \oplus \beta$ means 'identify the top face of α with the bottom face of β'). The new n-omino is self-avoiding by definition of the top face and the bottom face. Since we do not destroy any vertices on σ and ρ in this construction, concatenation is an injection $i : \mathcal{S}_n \times \mathcal{S}_m \hookrightarrow \mathcal{S}_{n+m}$ taking pairs $(\sigma, \rho) \mapsto (\sigma \oplus \rho)$. Therefore

$$s_n s_m \leq s_{n+m}. \tag{2.2}$$

Let $\sigma^g \in \mathcal{S}_n(P^g)$ and $\rho^h \in \mathcal{S}_m(P^h)$ where P is an extraordinary pattern. We want to concatenate σ^g and ρ^h in such a way that the number of patterns is additive. Consider the concatenated n-omino $\sigma^g \oplus \rho^h$. Since P is an extraordinary pattern the maximum number of times P can occur is $(g + h)$ times. If the top face of σ^g, or the bottom face of ρ^h, or both, are on a pattern, then the concatenated n-omino may have less than $(h + g)$ patterns P. The minimum number of patterns in the concatenated n-omino is $(g + h - 2)$. To deal with this possibility, select a k such that $s_k(P^i) > 0$ for $i = 0, 1, 2$. This is always possible since P is a proper pattern. Let $\omega_i^i \in \mathcal{S}_k(P^i)$ for $i = 0, 1, 2$. Since P is an extraordinary pattern, and therefore proper, it is always possible to choose the ω_i^i and k such that neither the top face, nor the bottom face are on a pattern P. Consider the concatenated n-omino $\sigma^g \oplus \omega_i^i \oplus \rho^h$. This n-omino is self-avoiding by definition of the top and bottom face, and we select i such that it contains the pattern P precisely $(g + h)$ times on its interface. Since we do not destroy any vertices on σ^g and ρ^h, this concatenation is an injection $i : \mathcal{S}_n(P^g) \times \mathcal{S}_m(P^h) \hookrightarrow \mathcal{S}_{n+m+k}(P^{g+h})$ such that $(\sigma^g, \rho^h) \mapsto (\sigma^g \oplus \omega_i^i \oplus \rho^h)$. Put $c = k$. Then

$$s_n(P^g) s_m(P^h) \leq s_{n+m+c}(P^{g+h}). \tag{2.3}$$

We take these results together in the following lemma.

Lemma 2.8. Let $d \geq 2$, and let P be an extraordinary pattern. Then there exists a constant c, dependent only on d and P, such that

$$s_n s_m \leq s_{n+m}$$
$$s_n(P^g) s_m(P^h) \leq s_{n+m+c}(P^{g+h}).$$

Taking lemmas 2.1 and 2.8 together we find the following result.

Proposition 2.9. Let $d \geq 2$, and let P be an extraordinary pattern. Then there exists constants $\beta_0(P)$ and β such that

$$\lim_{n\to\infty} s_n^{1/n} = \beta$$

and

$$\lim_{n\to\infty} s_n(P^0)^{1/n} = \beta_0(P).$$

Moreover, $s_n \leq \beta^n$, and there exists an integer c such that $s_n(P^0) \leq \beta_0(P)^{n+c}$ for all $n > 0$.

Proof. These results follows directly from the theory of subadditive functions (Hille 1948, Wilker and Whittington 1979) and lemmas 2.1 and 2.8. □

2.4. The construction of proper patterns

We are now in a position to prove an interesting fact about proper patterns. Let P and P' be two proper patterns on $\partial\sigma$, where $\sigma \in \mathcal{S}_n$. Then we say that P is *contained* in P' ($P \subset P'$) if there exists at least one subset of $(d-1)$- cells in P' which is identical to P.

Proposition 2.10.

(1) If P is a proper pattern, then there exists a proper pattern P' and an n-omino in the set $\mathcal{M}_2(P')$, where $P \subset P'$.

(2) If $\mathcal{M}_2(P) \neq \emptyset$, then there exists a proper pattern P' such that $P \subset P'$.

Proof.

(1) Suppose that P is a proper pattern. Then there exists a $\sigma \in M_1(P)$ such that the bottom face of σ is not on P. We have to consider two possible cases. (i) Suppose that P is expressed on the component of $\partial\sigma$ which separates the inside of σ from a point at infinity. Then we can assume that $\partial\sigma$ has only one component. Let P' be the pattern which contains every $(d-1)$-cell in $\partial\sigma$, except the bottom cell. Let ρ be the n- omino σ rotated 180° around a lattice axis perpendicular to e_1. Then $\partial\rho$ expresses P', and the top face of ρ is the only $(d-1)$-cell of $\partial\rho$ not in P'. Consider $(\rho \oplus \sigma)$. This n-omino contains P' exactly twice (since every $(d-1)$-cell on $\partial(\rho \oplus \sigma)$ is on P'). Therefore $(\rho \oplus \sigma) \in M_2(P')$, where $P \subset P'$. (ii) Suppose that P is expressed in a cavity inside σ. Then we consider $(\sigma \oplus \sigma)$, which contains two copies of the cavity that contains P. Choose P' to be the pattern containing every $(d-1)$-cell on the cavity. Then $\partial(\sigma \oplus \sigma)$ contains P' twice, and $P \subset P'$.

(2) Suppose that $\sigma \in M_2(P)$. Then there are two possible cases to consider. (i) Suppose that the bottom face of σ is in P. There are again two cases. (a) If the patterns are expressed on two different components of $\partial\sigma$, then $\partial\sigma$ has at least two components, one of which is a cavity. Choose P' to be every $(d-1)$-cell on the cavity which expresses P, then we have an n-omino containing P' once, and which has a bottom face not on P'. (b) If the patterns are on the same boundary component, then they must be on the component separating the inside from σ from a point at infinity (since the bottom face is on this component). Choose P' to be every face on this component, except for the bottom face. Then $\partial\sigma$ contains P' once and P' is proper. Moreover $P \subset P'$. (ii) Suppose that the bottom face of σ is not on P. There are again two cases. (a) If the patterns are on the same component of $\partial\sigma$, then let P' be the pattern containing every $(d-1)$-cell on that component, except for the bottom face. Then P' is a proper pattern containing P twice. (b) If the patterns are on different boundary components, then at least one must be expressed in a cavity. Fill this cavity in, and let $P' = P$ on the other component. Then P' is a proper pattern. □

We have defined proper and extraordinary patterns in section 2.2. In this section we prove an important inequality between the number of n-ominoes which lack a particular proper (or extraordinary) pattern and the number of n-ominoes which express the pattern precisely g times on its interface. Let p be a permutation on the integers $\mathcal{Z}_d = \{1, 2, \ldots, d\}$. Let $C_q = (v, a_1, a_2, \ldots, a_q)$ be a q-cell. Then there exists a permutation p such that $a_i = \pm e_{p(i)}$ for $1 \leq i \leq q$. We begin by considering the properties of some q- cells.

Definition 2.11. A hypercube $D_l(v)$ of size l is the subspace

$$\{x \in \mathcal{Z}^d \mid x = v + \sum_{i=1}^{d} c_i e_i,\ 0 \leq c_i \leq l,\ \forall i\}.$$

Definition 2.12. Let $c_i = \pm e_{p(i)}$ where $0 \leq i \leq q$ and where p is a permutation on $\mathcal{Z}_d$. Let $U_q = (v, c_1, c_2, \ldots, c_q)$ be a q-cell incident on two d-cells, $A = (v, a_1, a_2, \ldots, a_d)$ and $B = (v, b_1, b_2, \ldots, b_d)$, where $a_i = b_i = c_i$ if $1 \leq i \leq q$ and where $a_i = \pm e_{p(i)}$ and $b_i = \mp e_{p(i)}$ for $q < i \leq d$. If $a_i \neq b_i$ (have opposite sign) for $q < i \leq d$, and if $q < (d-1)$, then we call U_q *uncommon*. (In other words, if q is the largest integer such that U_q is incident on both A and B, and $q < (d-1)$, then U_q is uncommon.)

If U_q is uncommon, then it cannot be incident on more than two d-cells. This follows immediately from the fact that there are only two choices of the a_i and the b_i for each $i > q$ in definition 2.12. If U_q is incident on more than two d-cells, then U_q is a face of a $(q+1)$-cell C_{q+1} incident on at least two of the d-cells. If C_{q+1} is incident on precisely two d-cells, and $(q+1) < (d-1)$, then C_{q+1} is uncommon.

We can now prove the main result of this section.

Proposition 2.13. Let $d \geq 2$ and let $\sigma \in \mathcal{S}_n(P^0)$, where P is a proper pattern. Then it is possible to construct P in at least $\lfloor n/C \rfloor$ locations in σ, where C is a positive constant dependent only on d and P. Furthermore, there exists finite, positive constants K and k (dependent only on d and P) such that

$$\binom{\lfloor n/C \rfloor}{g} s_n(P^0) \leq K^g s_{n+kg}(P^g).$$

$\lfloor x \rfloor$ is the largest integer smaller or equal to x, x a real number.

Proof. Let $\lambda(P)$ be the size of P as defined in definition 2.5. Then there exists a $\rho \in \mathcal{S}_{n_0}(P^1)$ such that ρ has a bottom face not on P and such that ρ can be fitted into a hypercube of size $\lambda(P)$. Let $D_l(v)$ be a hypercube of size l. Then σ can be covered by at least $\lfloor n/l^d \rfloor$ copies of $D_l(v)$ for appropriate choices of v.

Choose any of the hypercubes, say D. Let D_i, $i = 1, 2$, be hypercubes with the same midpoints as D, but with sizes $(l-2)$ and $(l-4)$ respectively. This is illustrated in figure 5(a), which is a projection of σ and D onto the (e_i, e_j)-plane.

Consider the subspace $D - D_2$, which is inside the hypercube D, but outside the hypercube D_2. Occupy every location in this subspace not already occupied with a d-cell, and delete all the d-cells of σ inside D_2. This produces a new connected n-omino σ'. A projection on two dimensions is illustrated in figure 5(b). Some q-cells in σ' may be uncommon. These q-cells have to be in ∂D, the boundary of D, since we have added cells in the space $D - D_1$ while leaving σ unchanged outside D. We shall now remove these uncommon cells by deleting some q-cells in the subspace $D - D_1$.

Let u be an uncommon q-cell in σ'. Then by definition 2.12 u is incident on precisely two d-cells A and B, A in the section of σ' outside D and B in $D - D_1$. Since B has a face in ∂D, A has at most a q-cell in ∂D. If $q = (d-1)$, then A and B share a face on ∂D, which is not uncommon, therefore $q < (d-1)$. Suppose that $q = 0$. Then u is a vertex of B and therefore a vertex of D. Delete B. This action removes u as an uncommon cell, since u is then only incident on A. Moreover, deleting B in the

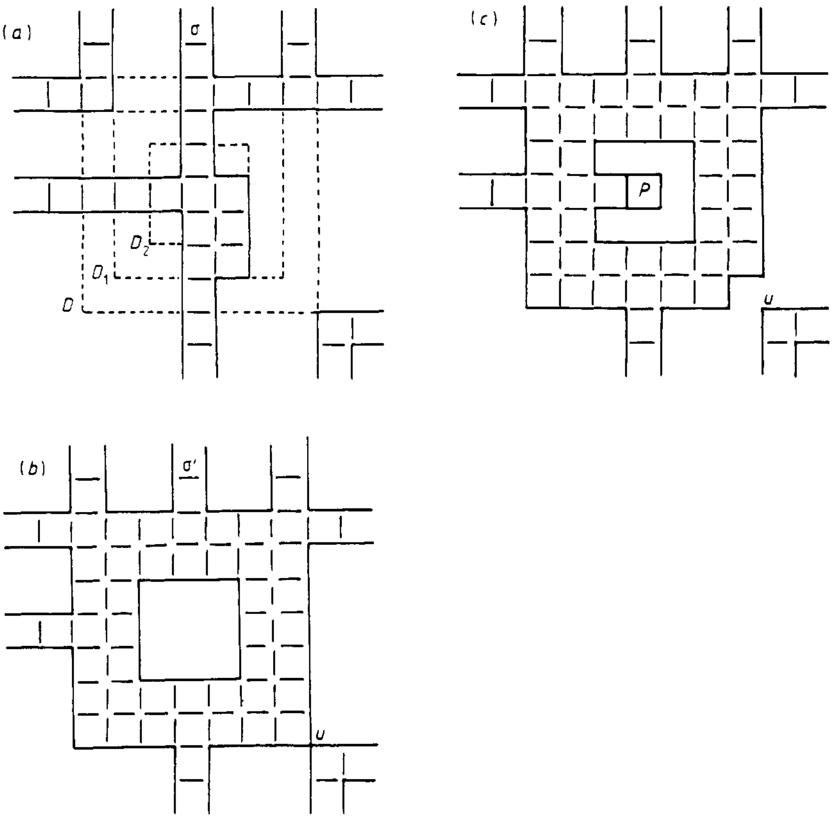

Figure 5. Constructing a new proper pattern in an n-omino. These diagrams are projections of the construction in d dimensions to the plane.

space $D - D_2$ does not create any uncommon cells or does not disconnect σ', since the subspace $D_1 - D_2$ is filled. Suppose that $q > 0$. Then we can again delete B. This will reduce u to at most $2q$ uncommon $(q-1)$-cells, since if A and B both contain u, then they also contain the boundary of u, which consists of $(q-1)$-cells. By induction it then follows that we can delete cells in the subspace $D - D_1$ to turn all uncommon q-cells common. A projection of this situation is illustrated in figure 5(c).

Lastly, if these deletions turn some q-cells in the space $D - D_1$ uncommon, then we repair that damage as follows. Suppose v is an uncommon q-cell in $D - D_1$. Then v is incident on two d-cells in $D - D_1$ (and $q > 1$). Deleting one of these d-cells will turn v common. Note that it is always possible to do this; we cannot disconnect σ'. Both d-cells incident on v have a face in ∂D. If both these faces are incident on d-cells in σ', then these cells must share an uncommon q-cell, which is a contradiction, since all the cells outside D are common.

The maximum number of ways that we can perform this construction is bounded by the number of ways that we can pack d-cells in D, say K_0. We choose $(l-6) = (\lambda(P)+2)$, then the cavity (the inside of D_3) is large enough to contain ρ. We add ρ to σ' by glueing its bottom face to the inside of the cavity with an extra d-cell. The maximum number of d-cells that we can remove from σ in this construction is l^d, and we may add as many as l^d d-cells to σ. This construction is therefore a map

$$\mathcal{S}_n(P^0) \to \bigcup_{j=n-l^d}^{n+l^d} \mathcal{S}'_j(P^1)$$

which is at most K_0 to 1 and where $\mathcal{S}'_j(P^1)$ is the set of all j-ominoes containing the pattern P once *inside* a cavity, and consists of j d-cells. Let the cardinality of $\mathcal{S}'_j(P^1)$ be $s'_j(P^1)$. Then, since the top face of any n-omino $\tau \in \mathcal{S}'_j(P^1)$ cannot be on the pattern P (P is inside a cavity), we have $s'_j(P^1) \leq s'_{j+1}(P^1)$. Hence

$$s_n(P^0) \leq K_0(2l^d+1)s'_{n+l^d}(P^1)$$

where $l = \lambda(P) + 8$. Obviously $s'_m(P^1) \leq s_m(P^1)$, so, if we put $K = K_0(2l^d+1)$ and $k = l^d$, and if we choose g hypercubes from $\lfloor n/C \rfloor$, where $C = l^d$, then we find

$$\binom{\lfloor n/C \rfloor}{g} s_n(P^0) \leq K^g s_{n+kg}(P^g).$$

□

3. Growth constants and patterns

Let P be a proper pattern consisting of p faces. Then the maximum number of times that P may be expressed on an interface $\partial\sigma$, where $\sigma \in \mathcal{S}_n$ is bounded by $\lceil 2dn/p \rceil$. Let $\mathcal{S}_n(P^{\lfloor \epsilon n \rfloor})$ be the set of all n-ominoes which express P precisely $\lfloor \epsilon n \rfloor$ times on their boundary components. Let the cardinality of this set be

$$s_n(\epsilon P) = s_n(P^{\lfloor \epsilon n \rfloor}). \tag{3.1}$$

Let $\epsilon_m^P = \max\{\epsilon \mid \sum_{n=0}^{\infty} s_n(\epsilon P) \neq 0\}$. Then $\epsilon_m^P \leq 2d/p$. Furthermore, let $\mathcal{S}_n(\leq \epsilon P)$ be the set of all n- ominoes which expresses P *at most* $\lfloor \epsilon n \rfloor$ times on their boundary components. The cardinality of this set is

$$s_n(\leq \epsilon P) = \sum_{g=0}^{\lfloor \epsilon n \rfloor} s_n(P^g). \tag{3.2}$$

Note that $s_n(\leq 0P) = s_n(P^0)$ and $s_n(\leq \epsilon_m^P P) = s_n$.

Lemma 3.1. Suppose that P is an extraordinary pattern. Then there exists a constant $\chi^P(\epsilon)$ such that

$$\chi^P(\epsilon) = \lim_{n\to\infty} s_n(\leq \epsilon P)^{1/n} \qquad \forall 0 \leq \epsilon \leq \epsilon_m^P.$$

Moreover, $\chi^P(\epsilon)$ is log-concave in $[0, \epsilon_m^P]$ and continuous in $(0, \epsilon_m^P)$.

Proof. To prove existence, note that equation (2.3) implies

$$s_n(\leq \epsilon P)s_m(\leq \epsilon P) \leq (\lfloor \epsilon n\rfloor + \lfloor \epsilon m\rfloor + 1)\sum_{g=0}^{\lfloor \epsilon n\rfloor+\lfloor \epsilon m\rfloor} s_{n+m+c}(P^g)$$
$$\leq (\lfloor \epsilon n\rfloor + \lfloor \epsilon m\rfloor + 1)s_{n+m+c}(\leq \epsilon P).$$

Therefore, the limit exists (Wilker and Whittington 1979). To prove log-concavity, let $0 \leq \delta_1, \delta_2 \leq \epsilon_m^P$. Then from equation (2.3) we find

$$s_n(\leq \delta_1 P)s_n(\leq \delta_2 P) \leq (\lfloor \delta_1 n\rfloor + \lfloor \delta_2\rfloor + 1)\sum_{g=0}^{\lfloor (2n+c)(\delta_1+\delta_2)/2\rfloor} s_{2n+c}(P^g)$$
$$= (\lfloor \delta_1 n\rfloor + \lfloor \delta_2 n\rfloor + 1)s_{2n+c}\left(\leq \frac{\delta_1+\delta_2}{2}P\right).$$

Taking logs, dividing by n and letting $n \to \infty$ gives

$$\log\chi^P(\delta_1) + \log\chi^P(\delta_2) \leq 2\log\chi^P\left(\frac{\delta_1+\delta_2}{2}\right)$$

and $\chi^P(\epsilon)$ is therefore log-concave in $[0, \epsilon_m^P]$. Continuity in $(0, \epsilon_m^P)$ follows from log-concavity. □

If P is a proper pattern, but not extraordinary, then we define

$$\chi^P(\epsilon) = \limsup_{n\to\infty} s_n(\leq \epsilon P)^{1/n}. \tag{3.3}$$

If P is an extraordinary pattern then $\chi^P(0) = \beta_0(P)$ and $\chi^P(\epsilon_m^P) = \beta$ in proposition 2.9. We can use the results from section 2 to prove the following inequality.

Proposition 3.2. If P is a proper pattern, then $\chi^P(0) < \chi^P(\epsilon)\ \forall\ 0 < \epsilon \leq \epsilon_m^P$.

Proof. Let $\delta > 0$ be any small real number. Then by proposition 2.9, lemma 3.1 and equation (3.3) there exists an infinite set $\mathcal{N}_0 \subset \mathcal{N}$ (where $\mathcal{N}$ is the set of natural numbers) such that for every $n \in \mathcal{N}_0$

$$(\chi^P(0) - \delta)^n \leq s_n(\leq 0P).$$

Consider now

$$s_n(\leq \epsilon P) = \sum_{g=0}^{\lfloor \epsilon n\rfloor} s_n(P^g)$$
$$\geq \sum_{g=0}^{\lfloor \epsilon n\rfloor} K^{-g}\binom{\lfloor (n-kg)/C\rfloor}{g} s_{n-kg}(\leq 0P) \qquad \text{by proposition 2.12}$$
$$\geq \sum_{g=0}^{\lfloor \epsilon n\rfloor} K^{-g}\binom{\lfloor (n-kg)/C\rfloor}{g}(\chi^P(0)-\delta)^{n-kg} \qquad \forall (n-kg) \in \mathcal{N}_0.$$

The combinatorial factor is 0 unless $g \leq \lfloor (n-kg)/C \rfloor$. Thus $g \leq \lfloor n/(C+k) \rfloor$. Choose $\epsilon < 1/(C+k)$, so that $\lfloor \epsilon n \rfloor \leq \lfloor n/(C+k) \rfloor$. Then $\lfloor \epsilon n \rfloor \leq \lfloor (n-kg)/C \rfloor$, and therefore

$$s_n(\leq \epsilon P) \geq \sum_{g=0}^{\lfloor \epsilon n \rfloor} K^{-g} \binom{\lfloor \epsilon n \rfloor}{g} (\chi^P(0) - \delta)^{n-kg}$$
$$= (\chi^P(0) - \delta)^n (1 + K^{-1}(\chi^P(0) - \delta)^{-k})^{\lfloor \epsilon n \rfloor}$$

for all $(n - kg) \in \mathcal{N}_0$ and $0 \leq \epsilon < 1/(C+k)$. Take the $1/n$ power, and let $n \to \infty$ in $\mathcal{N}_0 + kg$. Then we can take $\delta \to 0^+$, so that

$$\limsup_{n\to\infty} s_n(\leq \epsilon P)^{1/n} \geq \chi^P(0)(1 + K^{-1}\chi^P(0)^{-k})^{\epsilon}.$$

Therefore

$$\chi^P(\epsilon) > \chi^P(0) \qquad \forall 0 < \epsilon < 1/(C+k).$$

Lastly, note that $\chi^P(\epsilon)$ is monotone increasing in $[0, \epsilon_m^P]$, hence the above inequalility is valid in $(0, \epsilon_m^P]$. □

Let

$$\psi^P(\epsilon) = \limsup_{n\to\infty} s_n(\epsilon P)^{1/n}. \tag{3.4}$$

Then we have the following corollary.

Corollary 3.3. If P is a proper pattern, then there exists an ϵ in $(0, \epsilon_m^P]$ such that $\psi^P(0) < \psi^P(\epsilon)$.

Proof. Since $s_n(\leq \epsilon P) = \sum_{g=0}^{\lfloor \epsilon n \rfloor} s_n(P^g)$, $\chi^P(\epsilon)$ grows as fast as the term with the largest growth constant in equation (3.2). By proposition 3.2 $\chi^P(\epsilon) > \chi^P(0)$ if $\epsilon > 0$, and $\chi^P(0) = \psi^P(0)$. Hence, there exists an ϵ' in $(0, \epsilon]$ such that $\chi^P(\epsilon) = \psi^P(\epsilon')$. Hence $\psi^P(\epsilon') > \psi^P(0)$. □

We can now immediately prove our main result.

Theorem 3.4. Suppose that P is a proper pattern. Let $\langle N_n(P) \rangle$ be the mean number of times that P is expressed on n-ominoes in the set $\mathcal{S}_n$. Then

$$\liminf_{n\to\infty} \frac{\langle N_n(P) \rangle}{n} = \kappa > 0.$$

Proof. Consider

$$\langle N_n(P) \rangle = \frac{\sum_{g=0}^{\lfloor \epsilon_m^P n \rfloor} g s_n(P^g)}{\sum_{g=0}^{\lfloor \epsilon_m^P n \rfloor} s_n(P^g)}.$$

The denominator and numerator both have $\lfloor \epsilon_m^P n \rfloor$ terms, each term growing exponentially in n. The terms growing fastest are those with largest growth constants, which is $\psi^P(\epsilon)$ where $\epsilon > 0$ (by corollary 3.3). Choose κ' to be the smallest number such that $\psi^P(\kappa') = \psi^P(\epsilon)$. Then

$$\liminf_{n\to\infty} \frac{\langle N_n(P) \rangle}{n} = \liminf_{n\to\infty} \left(\frac{\sum_{g=0}^{\lfloor \epsilon_m^P n \rfloor} (g/n) s_n(P^g)}{\sum_{g=0}^{\lfloor \epsilon_m^P n \rfloor} s_n(P^g)} \right)$$
$$= \kappa \geq \kappa' > 0.$$

□

Evidently, we have (where P is a proper pattern)

$$0 < \kappa = \liminf_{n\to\infty} \frac{\langle N_n(P)\rangle}{n} \le \limsup_{m\to\infty} \frac{\langle N_n(P)\rangle}{n} \le \epsilon_m^P. \tag{3.5}$$

If we imply relation (3.5) with the relation $\sim$, then we may write

$$\langle N_n(P)\rangle \sim n. \tag{3.6}$$

By combining proposition 2.10 and theorem 3.4 we have the following.

Theorem 3.5. Suppose that P is a pattern which can occur twice on an n-omino in $\mathcal{S}_n$. Then

$$\langle N_n(P)\rangle \sim n.$$

Proof. By proposition 2.10 if $\mathcal{M}_2(P) \neq \emptyset$ then there exists a proper pattern $P' \supset P$. By theorem 3.4 $\langle N_n(P')\rangle \sim n$. Therefore $\langle N_n(P)\rangle \sim n$. □

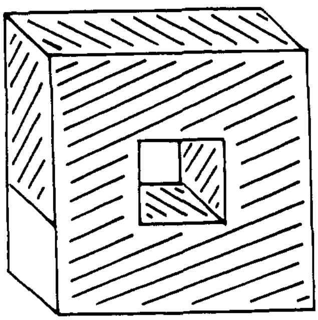

Figure 6. A pattern with genus one.

4. Discussion

(1) Let P be the pattern in figure 6. Then P is extraordinary, so by theorem 3.4 we know that $\langle N_n(P)\rangle \sim n$. But the genus of an interface of an n-omino is bounded (from below) by $N_n(P)$. Therefore, the genus of the average interface is $\langle g\rangle \sim n$. Let g represent any topological property which can be expressed on the interface of a finite n-omino in d dimensions. Then we expect the number of times this property is expressed on an n-omino to be proportional to the volume of the n-omino. We have the following theorem.

Theorem 4.1. Let $d \ge 2$. Let g be any topological property of a finite n-omino or of its embedding. Suppose that this property can occur more than once on an n-omino. Then the average number of times it occurs on an n-omino grows proportional to the number of d-cells in the n-omino, where proportional means in the sense of theorem 3.5.

(2) By theorem 4.1 the average surface area of an n-omino grows as its volume. This is an interesting result: the n-ominoes in $\mathcal{S}_n$ are ramified in the sense that each of the d-cells has a face on $\partial\sigma$, where $\sigma \in \mathcal{S}_n$, with positive probability.

(3) Some outstanding questions remain. Can we prove that $\lim_{n\to\infty}\langle N_n(P)\rangle/n = \kappa_P$ exists? What is the numerical value of κ_P?

(4) Can we prove that $\psi^P(\epsilon) = \lim_{n\to\infty} s_n(\epsilon P)^{1/n}$ (equation (3.1)) exists? Can we prove log-concavity and continuity of $\psi^P(\epsilon)$ and $\chi^P(\epsilon)$ in $[0,\epsilon_m^P]$? We were partially successful with $\chi^P(\epsilon)$, if P is an extraordinary pattern (lemma 3.1). For $\psi^P(\epsilon)$ we had to be satisfied with the definition in equation (3.4). Following the arguments developed by Madras *et al* (1988), we note that the following properties of $\psi^P(\epsilon)$ are important. Is $\psi^P(\epsilon)$ strictly concave? If it is not, then there is a first-order phase transition. If the maximum value in $\psi^P(\epsilon)$ is not attained at a unique point, then there is a transition, but at infinite temperature.

(5) There is a need to generalize this work to the embeddings of q-cells in d dimensions. If $q = 2$, then we have surfaces (I, II). Proving a pattern theorem for surfaces will be an important step in the understanding of these n-ominoes. This work is also closely related to the results of Madras (1989) on lattice animals. Every n-omino in $\mathcal{S}_n$ is dual to a site-animal, but the opposite is not true, there are animals which are not dual to n-ominoes. In particular, if we consider an n-omino dual to any animal, then it might contain uncommon q-cells.

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